Quadratic System of Two Differential Equations with Six Limit Cycles: Two Approaches to Problem Analysis

A. V. Voronin,
Department of Higher Mathematics and Economic and Mathematical Methods, Simon Kuznets Kharkiv National University of Economics, Ukraine, E-mail: voronin61@ukr.net
S. S. Lebedev
Department of Higher Mathematics and Economic and Mathematical Methods, Simon Kuznets Kharkiv National University of Economics, Ukraine, E-mail: Stepan.Lebedev@hneu.net

Abstract

In this paper, we propose two approaches to analyze the dynamic properties of a system of two differential equations with quadratic nonlinearity. It was demonstrated that both the method known from the literature and the method proposed by the authors of this paper give the same result; namely, in such nonlinear dynamic systems there are two foci and six limit cycles in the 3:3 arrangement.

Keywords

Differential equations with quadratic nonlinearity; Hilbert's sixteenth problem; singular points (the focus); limit cycles; Lyapunov quantities.

1. Introduction

One of the directions in the development of the theory of dynamical systems is the study of limit cycles. The second part of Hilbert's 16th problem is connected precisely with the determination of the possible number of limit cycles of polynomial vector fields and their location on the plane [1]. For quadratic systems of two differential equations, it has been proven that the number of limit cycles around a singular point (called the focus) cannot be more than three, see [2], [3], [4]. In the case of a quadratic system of differential equations for which there are two singular points, there is a specific example with four limit cycles in a 3:1 arrangement, i.e. three cycles around one focus and one cycle around another focus [5]. There is a research program, see [6], [7], [8] aimed at solving Hilbert's sixteenth problem for quadratic systems. However, until now the question of finding the upper bound on the number of limit cycles of quadratic systems remains open. In a previous paper [9], which was posted on ResearchGate in June 2022, we considered a quadratic system of differential equations with two focal points. In our antecedent study, the existence of 6 limit cycles in the arrangement 3:3 was proved. To verify this result, in this study we consider the same example of a system of two differential equations with quadratic nonlinearities that has two foci, but we use another method for studying it, which was proposed in [10] for the analysis of systems of this type.

2. Main results with examples

Consider a three-parameter system of two differential equations:

$$\begin{cases} \dfrac{dx}{dt} = \lambda x - y - (7+\varepsilon_1)x^2 + (5+\varepsilon_2)xy + 3y^2; \\ \dfrac{dy}{dt} = x + x^2 - 6xy. \end{cases} \tag{1}$$

This system (1) has two singular points (points of the focus type) with coordinates $x_1^* = 0, \quad y_1^* = 0$ and $x_2^* = 0, \quad y_2^* = 1/3$.

2.1. The first way of analysis

Let us pass to new variables $u = x, \quad v = y - y_{1,2}^*$ and study system (1) near singular points. After transformation we get:

$$\begin{cases} \dfrac{du}{dt} = (\lambda + (5+\varepsilon_2)y^*_{1,2})\cdot u + (6y^*_{1,2} - 1)\cdot v - (7+\varepsilon_1)\cdot u^2 + (5+\varepsilon_2)\cdot u\cdot v + 3v^2; \\ \dfrac{dv}{dt} = (1 - 6y^*_{1,2})\cdot u + u^2 - 6u\cdot v. \end{cases} \tag{2}$$

Than we introduce additional parameters: $S = 1 - 6y^*_{1,2}$ and $\mu = \frac{1}{2}\left(\lambda + (5+\varepsilon_2)y^*_{1,2}\right)$. This will allow us to describe the dynamic properties of system (2) in the vicinity of both singular points simultaneously. For the first singular point, which has coordinates $(0;0)$, we obtain the values of these parameters $S = 1$ and $\mu = \frac{1}{2}\lambda$, and for the second singular point $(0;1/3)$ we have $S = -1$ and $\mu = \frac{1}{2}\left(\lambda + \frac{5+\varepsilon_2}{3}\right)$ accordingly. In what follows, we assume that the parameters μ, ε_1 and ε_2 are small sign-alternating quantities. Taking into account the new notation, system (2) takes the form:

$$\begin{cases} \dfrac{du}{dt} = 2\mu u - Sv - (7+\varepsilon_1)\cdot u^2 + (5+\varepsilon_2)\cdot u\cdot v + 3v^2; \\ \dfrac{dv}{dt} = Su + u^2 - 6u\cdot v. \end{cases} \tag{3}$$

In this case, we check whether there exist for system (3) limit cycles around singular points (focal points). The characteristic polynomial for the linear part of system (3) is written as:

$$\begin{vmatrix} k-2\mu & S \\ -S & k \end{vmatrix} = 0 \quad \Rightarrow \quad k^2 - 2\mu k + S^2 = 0$$

Since $S^2 = 1$, then we get

$$k^2 - 2\mu k + 1 = 0. \tag{4}$$

Given that the parameter μ is small, characteristic equation (4) has the following roots:

$$k_{1,2} = \mu \pm i, \quad \text{where } i^2 = -1. \tag{5}$$

We differentiate equation (4) by the parameter μ. When $\mu = 0$ we get:

$$\frac{dk}{d\mu} = \frac{1}{2} \neq 0. \tag{6}$$

In this case, according to Hopf's theorem [11], [12], there are periodic solutions in a neighborhood of singular points.

We introduce new variables $u = x_1$, $v = S(\mu x_1 + y_1)$, and as a result of which we transform system (3) to the form of the Poincaré normal form [13]:

$$\begin{cases} \dfrac{dx_1}{dt} = \mu x_1 - y_1 - (7+\varepsilon_1 - 5S\mu)\cdot x_1^2 + S\cdot(5+\varepsilon_2+6S\mu)x_1\cdot y_1 + 3y_1^2; \\ \dfrac{dy_1}{dt} = x_1 + \mu y_1 + S(1-13S\mu)\cdot x_1^2 - (6+5S\mu)\cdot x_1\cdot y_1 - 3\mu y_1^2. \end{cases} \tag{7}$$

Let's employ complex conjugate variables $z = x_1 + i\cdot y_1$ and $\bar{z} = x_1 - i\cdot y_1$. With the help of these variables, we transform the system (7), containing two differential equations, to one complex differential equation with respect to the variable z:

$$z + g_{20}\frac{z^2}{2} + g_{11}z\cdot\bar{z} + g_{02}\frac{\bar{z}^2}{2}, \tag{8}$$

where

$$g_{20} = \frac{1}{2}\left(-16-\varepsilon_1 + i\cdot S(-4-\varepsilon_2 - 16S\mu)\right);$$
$$g_{11} = \frac{1}{2}\left(-4-\varepsilon_1 + 5S\mu + i\cdot S(1-16S\mu)\right);$$
$$g_{02} = \frac{1}{2}\left(-4-\varepsilon_1 + 10S\mu + i\cdot S(6+\varepsilon_2 - 4S\mu)\right).$$

For the differential equation (8), provided that $\mu = 0$, analytical expressions for the first three Lyapunov quantities $l_1,\ l_2,\ l_3$ are known. If $l_1 = l_2 = l_3 = 0$, a conservative dynamical system with an infinite number of periodic trajectories takes place, but there are not any limit cycles. According to [14[, [15], the formulas for the Lyapunov quantities have the following form:

1) $l_1 = -\frac{1}{2}\operatorname{Im}\left(g_{20}g_{11}\right);$

2) $l_1 = 0;\ l_2 = -\frac{1}{12}\operatorname{Im}\left(g_{20} - 4\bar{g}_{11}\right)\left(g_{20} + \bar{g}_{11}\right)\bar{g}_{11}g_{02}$; (9)

3) $l_1 = l_2 = 0;\ l_3 = -\frac{5}{64}\left(4|g_{11}|^2 - |g_{02}|^2\right)\operatorname{Im}\left(g_{20} + \bar{g}_{11}\right)\bar{g}_{11}^2 g_{02}$.

Analyzing relations (9), one can see that the cyclicity of a singular point of the complex (focus type) is determined by the simple formula:

$$g_{20} = r\cdot \bar{g}_{11}. \tag{10}$$

If r is a complex number, then there is a single limit cycle, i.e. $\operatorname{Im}\left(g_{20}g_{11}\right) \neq 0$. If r is a real number and $r \neq -1$ (conservative case) and also $r \neq 4$, then there are two limit cycles with different types of stability. If $r = 4$, then there are three limit cycles. Taking into account the fact that in the latter case $g_{20} = 4\cdot \bar{g}_{11}$, from expression (9) we derive the following formula for the Lyapunov variable l_3:

$$l_3 = -\frac{25}{64}\left(4|g_{11}|^2 - |g_{02}|^2\right)\operatorname{Im}(\bar{g}_{11}^3 g_{02}). \tag{11}$$

Without loss of generality, we set $\mu = \varepsilon_1 = \varepsilon_2 = 0$. Then

$$g_{20} = -8 - 2Si;\quad \bar{g}_{11} = -2 - \frac{Si}{2};\quad g_{02} = -2 + 3Si. \tag{12}$$

Let us carry out intermediate calculations of the expressions that are included in the equation (11):

$$|g_{11}|^2 = \frac{17}{4};\quad |g_{02}|^2 = 13;\quad \bar{g}_{11}^3 = -\frac{1}{8}\left(4+Si\right)^3 = -\frac{1}{8}\left(52+47Si\right);$$
$$\bar{g}_{11}^3\cdot g_{02} = -\frac{1}{8}(52+47Si)(-2+3Si) = -\frac{1}{8}(-245+62Si);\quad \operatorname{Im}\left(\bar{g}_{11}^3\cdot g_{02}\right) = -\frac{31S}{4}.$$

Taking into account the results obtained, we substitute the values (12) into formula (11) and obtain an expression for the third Lyapunov quantity:

$$l_3 = \frac{775S}{64} \neq 0. \tag{13}$$

It follows from formula (13) that at the equilibrium point $\left(0;0\right)$ at $S = 1$ we obtain $l_3 = \frac{775}{64} > 0$, and at the point $\left(0;1/3\right)$ at $S = -1$ we have $l_3 = -\frac{775}{64} < 0$.

Thus, for the same values of the parameters of the nonlinear part of system (1), in the vicinity of the point $(0;0)$ at $\lambda=0$ and in the vicinity of the point $(0;1/3)$ at $\lambda=-5/3$, there are six limit cycles in the arrangement 3:3.

2.2. The second way of analysis

We consider a system of two differential equations. Let's go back to the system, which is presented in the Poincaré normal form (7). Assume that the parameters are equal $\varepsilon_1=0$, $\varepsilon_2=0$ and $\mu=0$. Since $\mu=\frac{1}{2}\left(\lambda+5y_{1,2}^*\right)$, respectively $\lambda_{1,2}=-5y_{1,2}^*$. As a result, the system in this case has such form:

$$\begin{cases} \dfrac{dx_1}{dt}=-y_1-7x_1^2+5S\cdot x_1y_1+3y_1^2; \\ \dfrac{dy_1}{dt}= x_1+S\cdot x_1^2-6x_1y_1. \end{cases} \tag{14}$$

From the comparison of system (14) with the system which was considered in [10],

$$\begin{cases} \dfrac{dx}{dt}=-y+a_{20}x^2+a_{11}xy+a_{02}y^2; \\ \dfrac{dy}{dt}=x+b_{20}x^2+b_{11}xy+b_{02}y^2. \end{cases} \tag{15}$$

we can write down the values of the system parameters (the symbols) using the relations:

$$A=a_{20}+a_{02};\quad B=b_{20}+b_{02};\quad \alpha=a_{11}+2b_{02};\quad \beta=b_{11}+2a_{20};$$
$$\gamma=b_{20}A^3-(a_{20}-b_{11})A^2B+(b_{02}-a_{11})AB^2-a_{02}B^3;\quad \delta=a_{02}^2+b_{20}^2+a_{02}A+b_{20}B. \tag{16}$$

Then we have (up to a positive factor)

(i) $V_1=\alpha A-\beta B$;

(ii) $V_2=\left(\beta(5A-\beta)+\alpha(5B-\alpha)\right)\gamma,\quad if\quad V_1=0$

(iii) $V_3=(\beta A+\alpha B)\gamma\delta,\quad if\quad V_1=V_2=0$.

Consider the behavior of the system in the first equilibrium position:
$x_1^*=0;\ y_1^*=0;\ \lambda_1=-5y_1^*=0;\ S=1$. In this case, the system (14) takes the form:

$$\begin{cases} \dfrac{dx_1}{dt}=-y_1-7x_1^2+5x_1y_1+3y_1^2, \\ \dfrac{dy_1}{dt}=x_1+x_1^2-6x_1y_1, \end{cases}$$

where

$$a_{20}=-7;\quad a_{11}=5;\quad a_{02}=3;$$
$$b_{20}=1;\quad b_{11}=-6;\quad b_{02}=0.$$

Accordingly, we determine the values of the symbols (16):

$$A=-7+3=-4;\quad B=1+0=1;\quad \alpha=5+2\cdot 0=5;\quad \beta=-6+2\cdot(-7)=-20;$$
$$\gamma=1\cdot(-4)^3-(-7-(-6))\cdot(-4)^2\cdot 1+(0-5)\cdot(-4)\cdot 1^2-3\cdot 1^3=-31.$$
$$\delta=3^2+1^2+3\cdot(-4)+1\cdot 1=-1.$$

In such a case, we have

(i) $V_1=5\cdot(-4)-(-20)\cdot 1=0$.

Thus, we got that $V_1=0$. Then

(ii) $V_2=\left(-20(5\cdot(-4)-(-20))+5\cdot(5\cdot 1-5)\right)\cdot(-31)=0$

Since $V_1=V_2=0$, then

(iii) $V_3 = (-20 \cdot (-4) + 5 \cdot 1) \cdot (-31) \cdot (-1) = 2635 \neq 0$.

Since $V_1 = V_2 = 0$ and $V_3 \neq 0$, therefore we have three limit cycles around the point $(0; 0)$.

Consider the behavior of the system in the second equilibrium position:

$x_2^* = 0;\ y_2^* = 1/3;\ \lambda_2 = -5y_2^* = -5/3;\ S = -1$.

In this case, the system (14) takes the form:

$$\begin{cases} \dfrac{dx_1}{dt} = -y_1 - 7x_1^2 - 5x_1y_1 + 3y_1^2, \\ \dfrac{dy_1}{dt} = x_1 - x_1^2 - 6x_1y_1, \end{cases}$$

where

$$a_{20} = -7; \quad a_{11} = -5; \quad a_{02} = 3;$$
$$b_{20} = -1; \quad b_{11} = -6; \quad b_{02} = 0.$$

Accordingly, we determine the values of the symbols (16):

$$A = -7 + 3 = -4; \quad B = -1 + 0 = -1; \quad \alpha = -5 + 2 \cdot 0 = -5; \quad \beta = -6 + 2 \cdot (-7) = -20;$$
$$\gamma = -1 \cdot (-4)^3 - (-7 + 6) \cdot (-4)^2 \cdot (-1) + (0 - (-5)) \cdot (-4) \cdot (-1)^2 - 3 \cdot (-1)^3 = 31;$$
$$\delta = 3^2 + (-1)^2 + 3 \cdot (-4) + (-1) \cdot (-1) = -1.$$

In such a case, we have

(i) $V_1 = (-5) \cdot (-4) - (-20) \cdot (-1) = 0$.

Thus, we got that $V_1 = 0$. Then

(ii) $V_2 = \left(-20\left(5 \cdot (-4) + 20\right) - 5\left(5 \cdot (-1) + 5\right)\right) \cdot 31 = 0$.

Since $V_1 = V_2 = 0$, then

(iii) $V_3 = (-20 \cdot (-4) - 5 \cdot (-1)) \cdot 31 \cdot (-1) = -2635 \neq 0$.

Since $V_1 = V_2 = 0$ and $V_3 \neq 0$, therefore we have three limit cycles around the point $(0; 1/3)$.

3. Conclusion

As you can see, the application of both the first and second approaches to the analysis of possible solutions of the quadratic system of two differential equations gave the same result. It was shown that all the bifurcations that were considered in this example have a local character. This is due to the fact that when studying them, only certain, sufficiently small neighborhoods of a singular point or a multiple limit cycle are considered and, accordingly, sufficiently small neighborhoods of the system parameters. The final solution of Gilbert's sixteenth problem requires a complete qualitative study of the system as a whole, i.e. we need a global theory of bifurcations.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

References

[1]. Han M., Li C., Li J.
Limit cycles of planar polynomial vector fields
Scholarpedia, 5(8):9648 (2010)
doi:10.4249/scholarpedia.9648

[2]. Bamon R.
Quadratic Vector Fields in the Plane Have a Finite Number of Limit Cycles

Publications Mathématiques de l'I.H.É.S, 64 (1986), pp. 111-142.
[3]. Chicone C., Shafer D.S.
Separatrix and Limit Cycles of Quadratic Systems and Dulac's Theorems
Trans. Amer. Math. Soc., 278(2) (1983), pp.585-612
[4]. Coppel W.A.
Some Quadratic Systems with at Most One Limit Cycle
Dynamics Reported / U. Kirchgraber &H.O. Walther, New York : Wiley, vol. 2 (1988), pp. 61-68.
DOI https://doi.org/10.1007/978-3-322-96657-5
[5]. Chen L.S., Wang M.S.
The Relative Position, and the Number, of Limit Cycles of a Quadratic Differential System
Acta. Math. Sinica, 22(1979), pp. 751–758.
DOI： http://dx.doi.org/
[6]. Dumortier F., Roussarie R., Rousseau C.
Hilbert's 16th Problem for Quadratic Vector Fields
J. Differential Equations, 110 (1994), pp.86-133.
[7]. Dumortier F., El Morsalani M., Rousseau C.
Hilbert's 16th Problem for Quadratic Systems and Cyclicity of Elementary graphics
Nonlinearity, 9(5) (1996), pp. 1209-1261.
[8]. Gaiko V.A.
Global Bifurcation Theory and Hilbert's Sixteenth Problem
Springer Science+Business Media, LLC. (2003)
DOI http://dx.doi.org/10.1007/978-1-4419-9168-3
[9]. Voronin A.V., Lebedev S.S.
Example of a quadratic system of two differential equations with six limit cycles
https://www.researchgate.net/search.Search.html?query=Example+of+a+quadratic+system+of+two+differential+equations+with+six+limit+cycl&type=publication (June 2022)
[10]. Chow S.-N., Li C., Wang D.
Normal forms and bifurcation of planar vector fields
Cambridge University Press. New York (1994), pp 390-391
[11]. Hassard B.D., Wan Y.H.
Bifurcation Formulae Derived from Center Manifold Theory
Journal Math. Anal Appl., 63(1) (1978), pp. 297-312
[12]. Hassard B.D., Kazarinoff N.D., Wan Y.H.
Theory and Applications of Hopf Bifurcation
Cambridge University Press, Cambridge (1981)
[13]. Chow S.N., Li C., Wang D.
Normal Forms and Bifurcation of Planar Vector Fields
Cambridge : Cambridge University Press (1994).
DOI https://doi.org/10.1017/CBO9780511665639
[14]. Zoladek H.
Quadratic Systems with Center and their Perturbations
J. Differential Equations, 109(2) (1994), pp. 223-273.
DOI 10.1006/jdeq.1994.1049
[14]. Voronin A.V., Chernyshov S.I.
Cycles in Nonlinear Macroeconomics
arXiv:0706.1013v2 [nlin.AO] (2007)
https://arxiv.org/pdf/0706.1013.pdf