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THE CAUCHY FUNCTION FOR DIFFERENCE EQUATIONS

An interesting and alternative to the well-known approach is the method of finding partial solutions of linear differential equations (with constant and variable coefficients), which is to use Cauchy function $G(x,y)$ that determines (for example, a linear differential equation of second order with constant coefficients) as a solution of the following Cauchy problem:

$$\begin{cases} \dfrac{d^2G}{dx^2}+p\dfrac{dG}{dx}+qG=0 \\ G\big|_{x=z}=0 \quad \dfrac{dG}{dx}\Big|_{x=z}=1 \end{cases} \qquad (1)$$

Then the particular solutions of the inhomogeneous equation

$$y''+py'+qy=f(x) \qquad (2)$$

is $$y(x)=\int_{x_0}^{x} G(x,z)\,f(z)\,dz \qquad (3)$$

Indeed,

$$\frac{dy}{dx}=G(x,x)\,f(x)+\int_{x_0}^{x}\frac{dG(x,z)\,f(z)\,dz}{dx}=$$

$$=\int_{x_0}^{x}\frac{\partial G(x,z)\,f(z)\,dz}{\partial x}$$

$$\frac{d^2y}{dx^2}=\frac{dG(x,z)\,f(z)}{dx}\Big|_{x=z}+\int_{x_0}^{x}\frac{d^2G(x,z)\,f(z)\,dz}{dx^2}=$$

$$=f(x)+\int_{x_0}^{x}\frac{dG(x,z)\,f(z)\,dz}{dx^2}$$

Let's substitute $y(x)$ from (3), $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ into the equation (2) and due to (1) we get an identity.
Now we find $G(x,z)$ for the equation

$$\frac{d^2y}{dx^2}+w_0^2y=f(x)\ . \qquad (4)$$

We rewrite the equation in terms of function $G(x,y)$

$$\frac{d^2G}{dx^2}+w_0^2G=0$$

and we will get

$$G(x,z)=C_1(z)\sin w_0x+C_2(z)\cos w_0x\ .$$

As $G(x,x)=0$ => $C_1(z)\sin w_0z+C_2(z)\cos w_0z=0$.

From the initial condition $\frac{dG}{dx}\big|_{x=z}=1$ we have

$$C_1(z)\cos w_0z-C_2(z)\sin w_0z=1\ .$$

Solving the system of this algebraic equations, we find

$$C_1(z)=\frac{\cos w_0z}{w_0} \quad \text{and} \quad C_2(z)=-\frac{\sin w_0z}{w_0}\ .$$

Hence, $G(x,z)=\frac{1}{w_0}\sin w_0(x-z)$.

Thus, a particular solution of the equation (4) is

$$y=\frac{1}{w_0}\int_{x_0}^{x}\sin w_0(x-z)\,f(z)\,dz\ .$$

For **inhomogeneous linear differential equations n-th order**

$$\sum_{k=0}^{n}a_k(x)\frac{d^ky}{dx^k}=f(x) \qquad (5)$$

Cauchy function $G(x,z)$ introduced similar in the sense that $G(x,z)$ is the solution of the following problem with appropriate initial data:

$$\sum_{k=0}^{n}a_k(x)\frac{d^kG(x,z)}{dx^k}=0$$

$$G\big|_{x=z}=0\,,$$

$$\frac{dG}{dx}\Big|_{x=z}=0\,,\ldots\ldots\frac{d^{n-2}G}{dx^{n-2}}\Big|_{x=z}=0\,,\ \frac{d^{n-1}G}{dx^{n-1}}\Big|_{x=z}=1\,.$$

Then the particular solution of the inhomogeneous equation (5) has the appearance $y(x)=\int_{x_0}^{x} G(x,z)\,f(z)\,dz$.

As for linear differential equations, difference equations if you can enter the Cauchy function, with which there is a particular solution of the inhomogeneous difference equation for arbitrary right side. Consider the simplest case of a linear difference equation of 1st order with constant coefficients

$$y_{n+1}+py_n=f(n) \qquad (6)$$

Let's introduce the Cauchy function as a solution to the Cauchy problem:

$$\begin{cases} G(n+1,m)+pG(n,m)=0 \\ G(n,m)\big|_{n=m}=1 \end{cases}.$$

The solution is $G(n,m)=(-p)^{n-m}=G(n-m)$. Then a particular solution of equation (6) for any $f(n)$ is $y_{n,\,particular}=\sum_{m=1}^{n-1} G(n-m-1)f(m)$.

Let as consider also **the inhomogeneous difference equation of the 2nd order with constant coefficients**

$$y'_{n+2}+py_{n+1}+qy_n=f(n) \qquad (7)$$

We find a partial solution of this equation using the Cauchy function $G(n,m)$, which is the solution of the following initial value problem:

$$G(n+2,m)+pG(n+1,m)+qG(n,m)=0;$$

$$G\big|_{n=m}=0 \qquad G(n+1,m)\big|_{n=m}=1$$

The corresponding characteristic equation has the form $z^2+pz+q=0$. Let, for example, be the roots of this equation λ_1 and λ_2 are real and different then

$$G(n,m)=C_1(m)\lambda_1^n+C_2(m)\lambda_2^m.$$

Then from the initial conditions

$$\begin{cases} C_1(m)\lambda_1^m+C_2(m)\lambda_2^m=0 \\ C_1(m)\lambda_1^{m+1}+C_2(m)\lambda_2^{m+1}=1 \end{cases}$$

The solution to this system is

$$G(n,m)=\frac{\lambda_1^{n-m}}{\lambda_1-\lambda_2}-\frac{\lambda_2^{n-m}}{\lambda_1-\lambda_2}=G(n-m).$$

Then the particular solution of equation (7) has the form

$$y_{n,\,particular}=\sum_{m=1}^{n-1} G(n-1-m)\,f(m)= \\ =\frac{1}{\lambda_1-\lambda_2}\sum_{m=1}^{n-1}\left(\lambda_1^{n-m-1}-\lambda_2^{n-m-1}\right)f(m) \qquad (8)$$

For **linear inhomogeneous difference equation of k -th order** type

$$\sum_{j=0}^{k} a_j y_{n+j}=f(n)$$

the Cauchy function $G(n,m)$ introduced as a solution corresponding equation with special initial data:

$$\sum_{j=0}^{k} a_j G(n+j,m)=0$$

$$G(n,m)\big|_{n=m}=0$$

$$\dots$$

$$G(n+k-2,m)\big|_{n=m}=0$$

$$G(n+k-1,m)\big|_{n=m}=1$$

Then the partial solution of the inhomogeneous equation is written as

$$y_n=\sum_{m=1}^{n-1} G(n-1,m)\,f(m)$$

Obviously, the same way you can get a solution in the case of multiple or complex roots of the characteristic equation.

References

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