Local vanishing property of solutions for parabolic PDEs

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Investigations are devoted to the study of the extinction of solutions in finite time to initial-boundary value problems for a wide classes of nonlinear parabolic equations of the second and higher orders with a degenerate absorption potential, whose presence plays a significant role for the mentioned nonlinear phenomena. As well known the extinction property means that any solution of the mentioned equation vanishes in Ω in a finite time, i.e. $\exists\ 0 < T_0 < \infty :\ u(t,x) = 0$ a.e. in $\Omega\ \ \forall\ t \geq T_0$.

So, we investigate a model Cauchy-Neumann problem for parabolic equations of non-stationary diffusion-semilinear absorption with a degenerate absorption potential. More precisely, the following problem is considered:

$$\left(|u|^{q-1}u\right)_t - \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(|\nabla_x u|^{q-1}\frac{\partial u}{\partial x_i}\right) + a_0(x)|u|^{\lambda-1}u = 0 \quad \text{in} \quad \Omega \times (0,T), \tag{1}$$

$$\frac{\partial u}{\partial n}\Big|_{\partial\Omega\times[0,T]} = 0, \tag{2}$$

$$u(x,0) = u_0(x), \quad x \in \Omega. \tag{3}$$

Here $q > 1$, $0 < \lambda < 1$, and $a_0(x) \geq 0$ is an arbitrary continuous function. The initial function $u_0(x)$ is from $L_{q+1}(\Omega)$, where $\Omega \subset \mathbb{R}^N (N \geqslant 1)$ is a bounded domain with C^1 - boundary. The origin belongs to Ω $(0 \in \Omega)$. The considered problem (1), (2), (3) has energy solution. It is follows from paper [1].

We obtain a sharp condition on the degeneration of the potential $a_0(x)$ that guarantees the long-time extinction. Let $a_0(x)$ be a potential satisfying the inequality

$$a_0(x) \geq c_0 \exp\Big(-\frac{\omega(|x|)}{|x|^{q+1}}\Big), \quad x \in \Omega \setminus \{0\}, \tag{4}$$

where $c_0 > 0$ is a constant, and $\omega(\cdot)$ is an arbitrary function such that

$$(A) \quad \omega(\tau) > 0\ \ \forall\tau > 0, \qquad (B) \quad \omega(0) = 0, \qquad (C) \quad \omega(\tau) \to 0 \text{ as } \tau \to 0 \text{ monotone.}$$

Let us formulate the main result of the work [2].

Theorem. *Let $u_0(x) \in L_2(\Omega)$. Let $\omega(\cdot)$ be a continuous nondecreasing function that satisfies assumptions (A), (B), (C) and the following main condition:*

$$\int_0^c \frac{\omega(\tau)}{\tau} d\tau < \infty. \tag{5}$$

Suppose also that $\omega(\cdot)$ satisfies the technical condition

$$\frac{\tau\, \omega'(\tau)}{\omega(\tau)} \leq 1 - \delta \quad \forall\ \tau \in (0, \tau_0), \quad \tau_0 > 0, \quad 0 < \delta < 1.$$

Then an arbitrary energy solution $u(x,t)$ of problem (1), (2), (3) vanishes on Ω in a finite time $T < \infty$.

Ideas of the proof is based on the local energy method in the spirit of papers [3], [4], [5].

Also we investigate the local vanishing property in the finite time of solutions to the initial-boundary problem for $2m$ order nonlinear parabolic equation with absorption of the following type problem:

$$\left(|u|^{q-1}u\right)_t + (-1)^m \sum_{|\eta|=m} D_x^\eta \left(|D_x^m u|^{q-1} D_x^\eta u \right) + a(x)|u|^{\lambda-1}u = 0 \ \text{ in } \ Q, \tag{6}$$

$$D_x^\eta u\Big|_{(0,+\infty)\times\partial\Omega} = 0 \qquad \forall \eta : |\eta| \le m-1, \tag{7}$$

$$u(0,x) = u_0(x), \qquad x \in \Omega. \tag{8}$$

where $Q = (0,+\infty)\times\Omega$, $\Omega \subseteq \mathbb{R}^N, N \geqslant 1$, $m \geq 1, 0 \leq \lambda < q$, an absorption potential $a(x)$ is nonnegative, measurable, bounded in Ω function.

Modifying the local energy approach of [6], we obtain sufficient conditions, which guarantee the extinction for the mentioned equation above. These conditions are depending on N, m, and on the parameter of homogeneous nonlinearity of the main part in the equation q:

$$\int_0^c \frac{(meas\{x\in\Omega : \ a(x) \le s\})^\theta}{s} ds < +\infty \qquad \forall\, c>0, \text{ where } \theta = \min\left(\frac{m(q+1)}{N}, 1\right), \ N \ne 2m, \tag{9}$$

$$\int_0^c \frac{meas\{x\in\Omega : \ a(x)\le s\}(-ln\, meas\{x\in\Omega : \ a(x)\le s\})}{s} ds < +\infty \ \ \forall\, c>0, \text{ for } N = m(q+1). \tag{10}$$

Main results of the paper [7] are the following:

Theorem.
If $N \ne 2m$ and (9) holds, then any solution of problem (6), (7), (8) has the extinction in finite time.
If $N = 2m$ and (10) holds, then any solutions of problem (6), (7), (8) has the extinction in finite time.

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